

# Towards a Curry-Howard Correspondence for Quantum Computation

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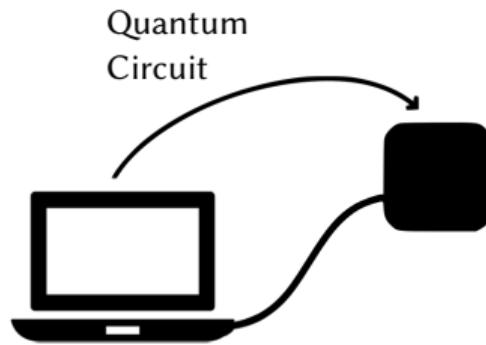
PhD Defense, 09/01/2023

Supervised by: Pablo Arrighi, Alexis Saurin, Benoît Valiron

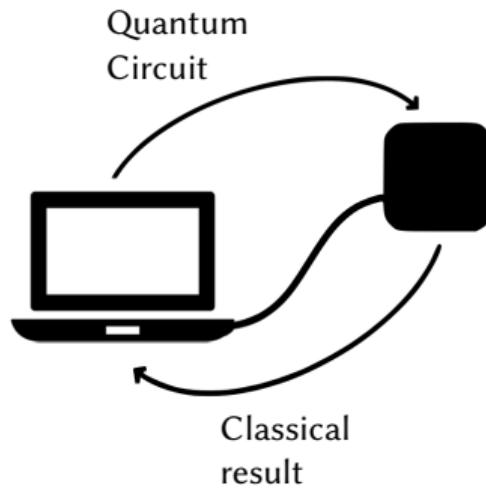
# The Coprocessor Model

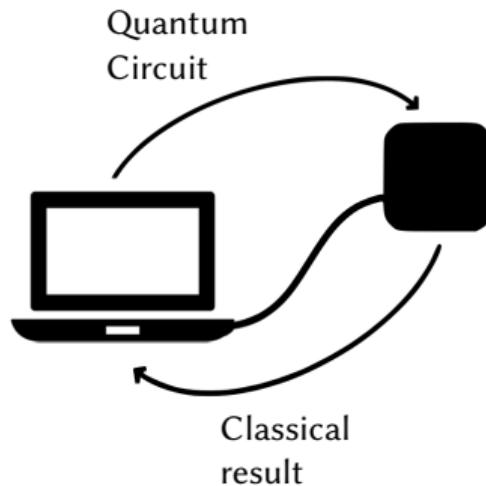


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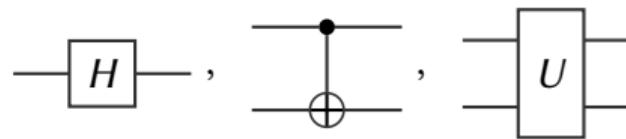
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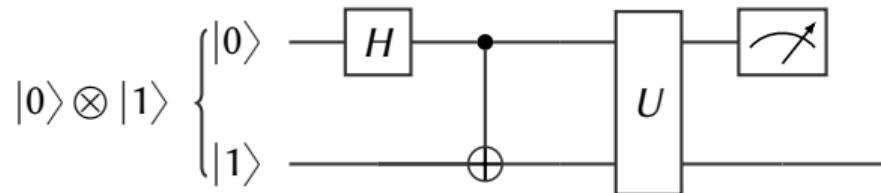


## Two operations

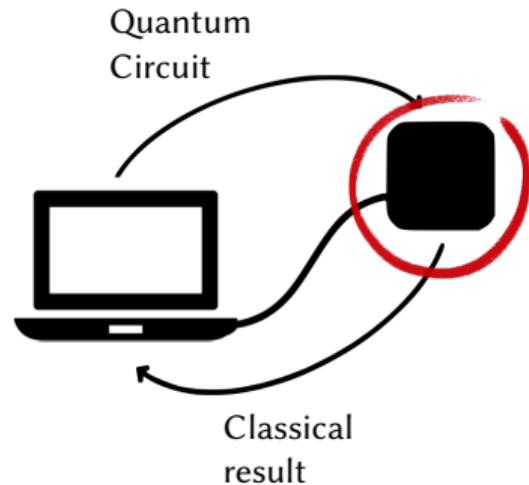
- Unitary operations **inside** the coprocessor.



- Probabilistic operation with **measure**.

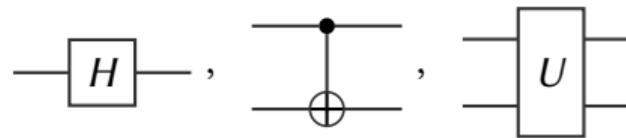


- Available data **inside coprocessor** :  $\otimes^n$  qubit.

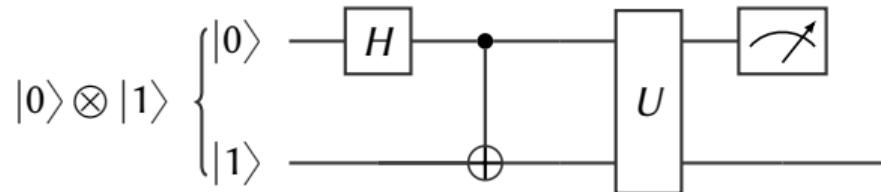


## Two operations

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- Probabilistic operation with **measure**.



- Available data **inside coprocessor** :  $\otimes^n$  qubit.

# Qubits and Quantum Operations

Classical	Quantum
0	$ 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
1	$ 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(0, 1)$	$ 0\rangle \otimes  1\rangle$

$$\alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

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$\alpha 0\rangle + \beta 1\rangle$	$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

- **Unitary Operations:** can be reversed.

- CNOT:

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \text{---} \\ \oplus \end{array} = \begin{cases} |0\rangle \otimes |x\rangle \mapsto |0\rangle \otimes |x\rangle \\ |1\rangle \otimes |x\rangle \mapsto |1\rangle \otimes \neg|x\rangle \end{cases}$$

- Hadamard:

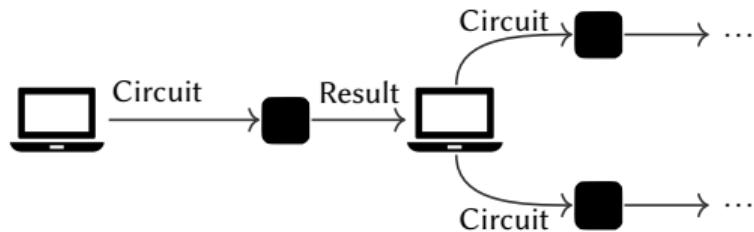
$$\begin{array}{c} \text{---} \\ \boxed{H} \\ \text{---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$$

- **Non-Cloning Principle.**



# From Classical to Quantum Control Flow

## Classical Control Flow

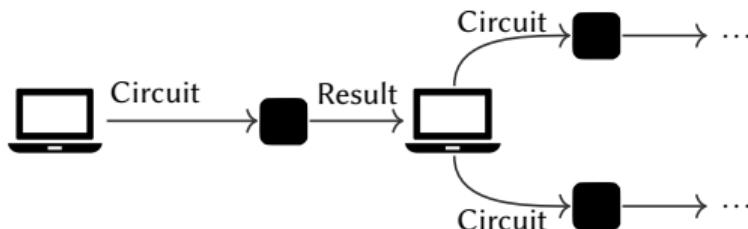


# From Classical to Quantum Control Flow

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## Classical Control Flow

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## Quantum Control Flow

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$$\text{QSwitch}(x, y, U, V) = \begin{cases} VU(y) & \text{if } x = |0\rangle \\ UV(y) & \text{if } x = |1\rangle \end{cases}$$

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |y\rangle \mapsto \alpha|0\rangle \otimes (UV|y\rangle) + \beta|1\rangle \otimes (VU|y\rangle).$$

Physically implementation but **not in co-processor.**

## Classical

- Bit =  $1 \oplus 1$ .

## QRAM

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## This thesis

Develop a new model of quantum computation featuring

- **A richer type system** (inductive);
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Develop a new model of quantum computation featuring

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**Approach :** Curry-Howard Correspondence.

## Types

- Type = Description of a data.  
product, choice, functions, ...
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- Study of **mathematical reasoning**.
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Types	$\leftrightarrow$	Propositions
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$$\frac{\begin{array}{c} \pi \\ \vdots \\ A \rightarrow B \end{array} \quad \begin{array}{c} \pi' \\ \vdots \\ A \end{array}}{B} \text{ Modus Ponens}$$

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## Formulas

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**No duplication or erasure.**

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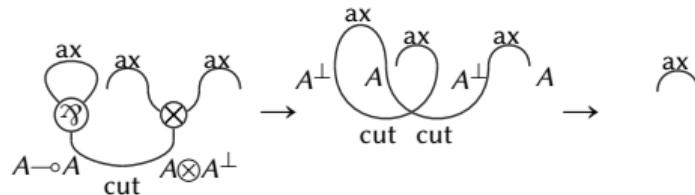
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## Proof Nets

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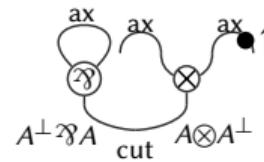
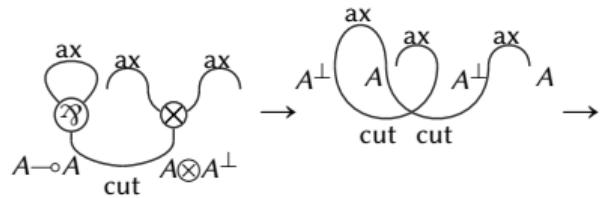
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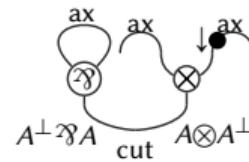
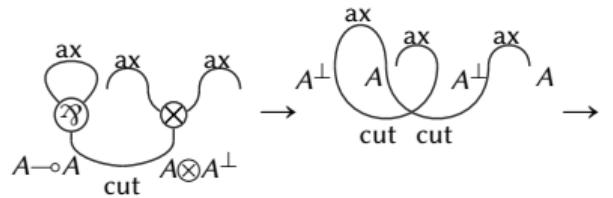
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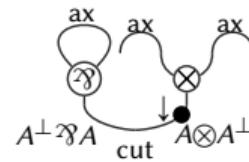
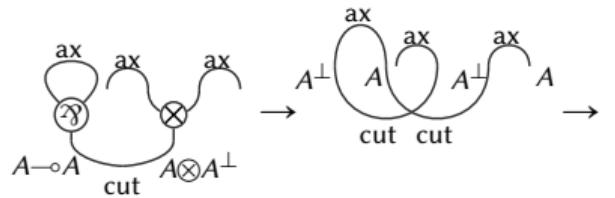
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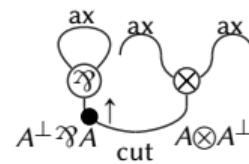
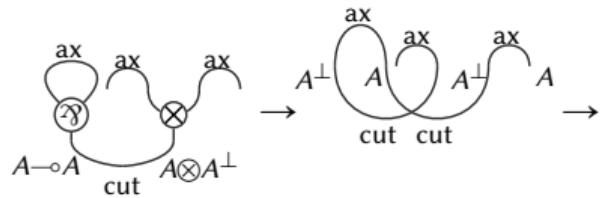
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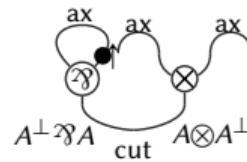
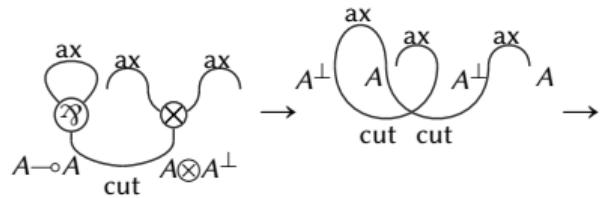
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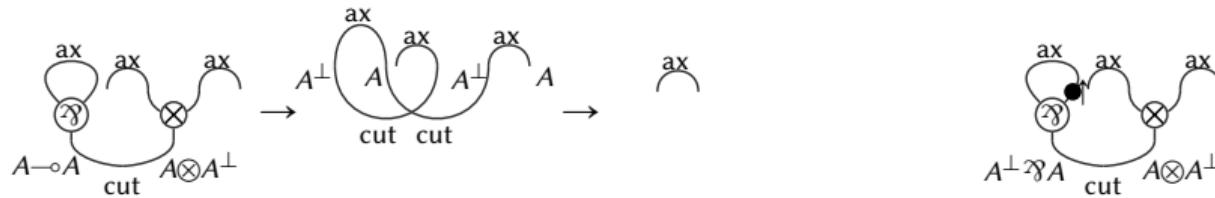
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## Proof Nets

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## Additives

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$$A, B ::= \dots \mid A \oplus B$$

$\text{Bool} = \mathbb{1} \oplus \mathbb{1}.$

$\oplus$  represent the **action of a choice**.

Two routes for quantum types.

## Classical Control

- $\text{bit} = \mathbb{1} \oplus \mathbb{1}$
- Allow duplication in a controlled way.
- Quantum  $\lambda$ -calculus [Selinger, Valiron'04].
- Classical control.

## Quantum Control

- $\text{qubit} = \mathbb{1} \oplus \mathbb{1}$
- Inductive types  
 $\text{list}(A) = \mu X.\mathbb{1} \oplus (A \otimes X)$ .
- Quantum Switch.

**Our proposal:** logic  $\mu\text{MALL}$ , MALL + least and greatest fixed-point.

# Towards a Curry-Howard Correspondence for Quantum Computation

$$[A] = \mu X. \mathbb{1} \oplus (A \otimes X)$$

l.f.p operator  
(Ch. 4, CSL'23)

Pairing  
(Ch. 5, MFCS'21)

Quantum Control  
(Ch. 6, Draft)

```
graph TD; A["[A] = μX. 1 ⊕ (A ⊗ X)"] --> B["l.f.p operator  
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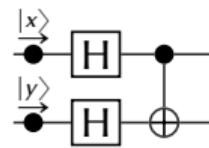
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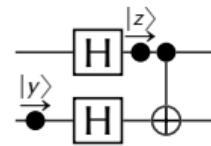
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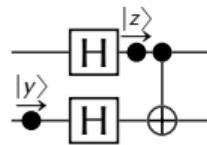
# Token Machines for Quantum Computation



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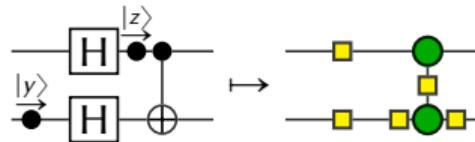


# Token Machines for Quantum Computation



- MELL + Circuits [Dal Lago. et al'16].
- Require **synchronisation**.
- No superposition of **position**.
- Classical Control.

# Token Machines for Quantum Computation

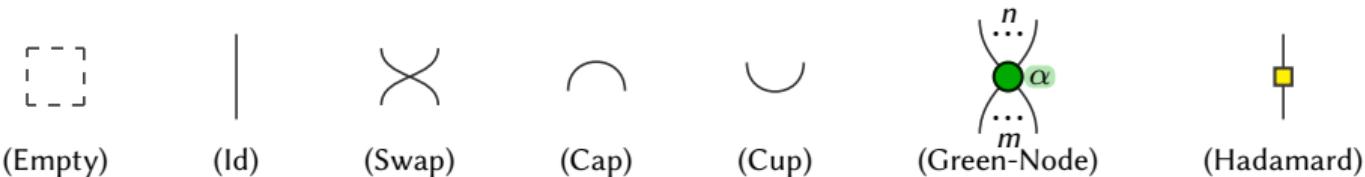


- MELL + Circuits [Dal Lago. et al'16].
- Require **synchronisation**.
- No superposition of **position**.
- Classical Control.
- Consider **Token Machine**.
- **Asynchronicity**.
- **Quantum Control**.

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Generators

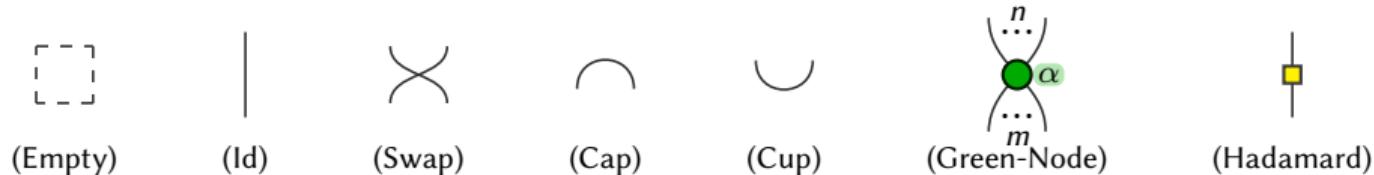
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Generators

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Compositions

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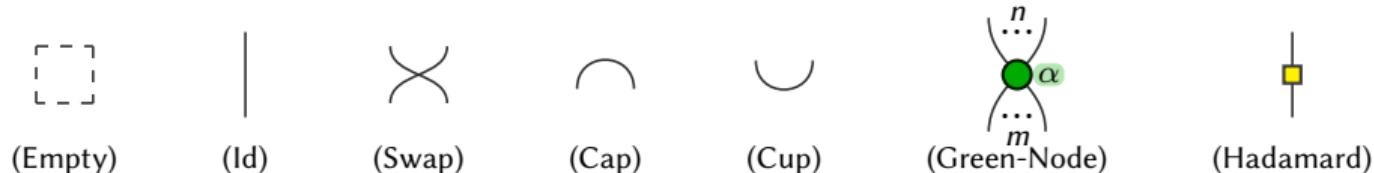
$$\begin{array}{c} \text{---} \\ | \quad | \\ \boxed{D_2} \quad \circ \quad \boxed{D_1} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \boxed{D_1} \\ \text{---} \\ | \quad | \\ \boxed{D_2} \end{array}$$

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## Generators

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## Compositions

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$$D_2 \circ D_1 = \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ D_2 \\ | \\ \dots \end{array}$$

$$D_1 \otimes D_2 = \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \\ | \\ D_2 \\ | \\ \dots \end{array}$$

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## Standard Interpretation

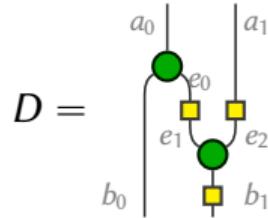
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Linear Maps :  $\mathbf{ZX} \rightarrow \mathcal{M}(\mathbb{C})$

A diagram showing a linear map interpretation. It consists of two parts: a diagram on the left and an equals sign followed by a matrix on the right.

The diagram on the left shows a green node with three outgoing lines. The top line passes through a yellow Hadamard gate. The middle line passes through another yellow Hadamard gate. The bottom line passes through a third yellow Hadamard gate. This corresponds to the matrix:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



## Token

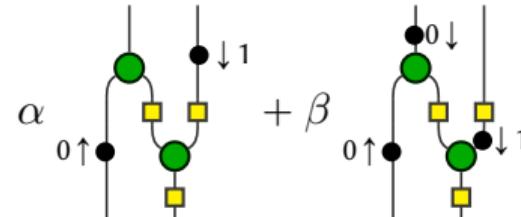
3-tuple  $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$  where:

- $e$  is an edge of the ZX-Diagram  $D$ .
- $d$  is a direction.
- $b$  is the state of the token.

## Token State

A *token state* is a **sum of products** of tokens with complex coefficients.

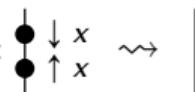
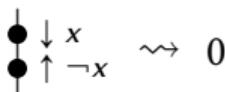
$$\langle t | t' \rangle = \begin{cases} 1 & \text{if } t = t' \\ 0 & \text{if } t \neq t' \end{cases}$$

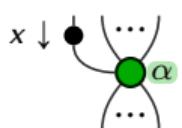
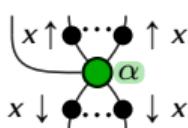


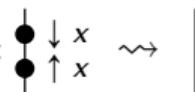
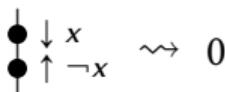
- Collisions :   $\rightsquigarrow$  

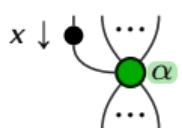
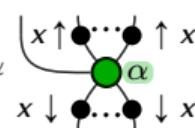
- Collisions :   $\rightsquigarrow$

$$\begin{array}{c} \bullet \\ \bullet \end{array} \downarrow \begin{array}{l} x \\ -x \end{array} \rightsquigarrow 0$$

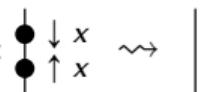
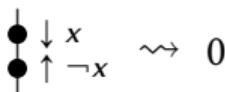
- Collisions :   $\rightsquigarrow$  |   $\rightsquigarrow 0$

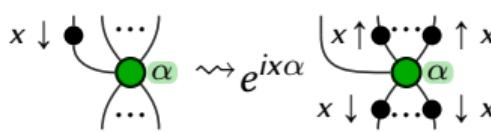
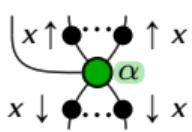
- Diffusions :   $\rightsquigarrow e^{ix\alpha}$  

- Collisions :   $\rightsquigarrow$  |   $\rightsquigarrow 0$

- Diffusions :   $\rightsquigarrow e^{ix\alpha}$  

$$\begin{array}{c} \bullet \\ \blacksquare \end{array} \downarrow x \rightsquigarrow \frac{1}{\sqrt{2}} \left( \begin{array}{c} \blacksquare \\ \bullet \end{array} \downarrow_0 + (-1)^x \begin{array}{c} \blacksquare \\ \bullet \end{array} \downarrow_1 \right)$$

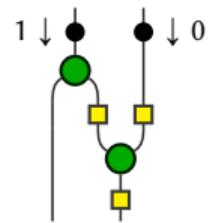
- Collisions :   $\rightsquigarrow$  |   $\rightsquigarrow 0$

- Diffusions :   $\rightsquigarrow e^{ix\alpha}$  
-   $\downarrow x$   $\rightsquigarrow \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{yellow square} \\ \text{black dot} \end{array} \downarrow 0 + (-1)^x \begin{array}{c} \text{yellow square} \\ \text{black dot} \end{array} \downarrow 1 \right)$
-   $\downarrow x$   $\rightsquigarrow \begin{array}{c} \text{black dot} \\ \curvearrowleft \end{array} \rightsquigarrow \begin{array}{c} \text{black dot} \\ \curvearrowright \end{array} \uparrow x$  ...

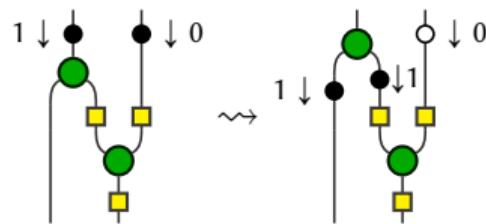
$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top wire and target on bottom wire. Control wire has a green circle at top and a yellow square at bottom. Target wire has a yellow square at top and a green circle at bottom. There is a curved arrow from the control circle to the target square.} \\ \text{---|---} \\ | \quad | \\ \text{---|---} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top wire and target on bottom wire. The control wire passes through a green circle (control) and a yellow square (not). The target wire passes through a yellow square (not) and a green circle (target).} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top wire and target on bottom wire.} \\ \text{Control wire (top)} \quad \text{Target wire (bottom)} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Initial State: } |1\rangle \downarrow \text{ and } |0\rangle \downarrow \\ \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} \rightsquigarrow^* \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{Two separate paths} \\ \text{Each path has a CNOT gate} \\ \text{Final states: } |1\rangle \downarrow |1\rangle \downarrow \text{ and } |0\rangle \downarrow |1\rangle \downarrow \end{array} \right)$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a square symbol on the control wire.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a square symbol on the control wire.} \\ \xrightarrow{\sim *} \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a square symbol on the control wire.} \\ + \text{Diagram of a CNOT gate with control on top and target on bottom, with a square symbol on the control wire.} \end{array} \right) \end{array}$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} \rightsquigarrow^* \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} + \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} \right)$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}}|11\rangle$$

$$\begin{array}{c} \text{Diagram of a multi-qubit circuit with two controls and one target, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \\ \rightsquigarrow^* \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \left( \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \\ - \end{array} \right) + \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \\ + \end{array} \right) \end{array}$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}}|11\rangle$$

$$\begin{array}{c} \text{Diagram of a 2-qubit gate with controls } 1 \downarrow \text{ and } 0 \downarrow, \text{ followed by a } \rightsquigarrow^* \text{ symbol and a } \frac{1}{2} \text{ coefficient.} \\ \rightsquigarrow^* * \frac{1}{2} \left( \begin{array}{c} \text{Four terms representing the expansion of the gate:} \\ - \text{Term 1: Diagram with controls } 1 \downarrow \text{ and } 0 \downarrow, \text{ target } 0 \downarrow \text{ with a } 0 \uparrow \text{ arrow from target } 0 \text{ to control } 1. \\ + \text{Term 2: Diagram with controls } 1 \downarrow \text{ and } 0 \downarrow, \text{ target } 0 \downarrow \text{ with a } 1 \uparrow \text{ arrow from target } 0 \text{ to control } 1. \\ - \text{Term 3: Diagram with controls } 1 \downarrow \text{ and } 0 \downarrow, \text{ target } 1 \downarrow \text{ with a } 0 \uparrow \text{ arrow from target } 1 \text{ to control } 1. \\ - \text{Term 4: Diagram with controls } 1 \downarrow \text{ and } 0 \downarrow, \text{ target } 1 \downarrow \text{ with a } 1 \uparrow \text{ arrow from target } 1 \text{ to control } 1. \end{array} \right) \end{array}$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}}|11\rangle$$

$$\begin{array}{c} \text{Diagram of a multi-qubit circuit with two controls and one target, with a } \frac{1}{2} \text{ coefficient.} \end{array} \rightsquigarrow * \frac{1}{2} \left( \begin{array}{c} \text{Diagram 1: Two controls (top 1, bottom 0) and one target (bottom 0).} \\ \text{Diagram 2: Two controls (top 1, bottom 0) and one target (bottom 1).} \\ \text{Diagram 3: Two controls (top 1, bottom 1) and one target (bottom 0).} \\ \text{Diagram 4: Two controls (top 1, bottom 1) and one target (bottom 1).} \end{array} - + - \right)$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}}|11\rangle$$

$$\begin{array}{c} \text{Diagram of a multi-qubit circuit with two controls and one target, followed by a } \rightsquigarrow^* \text{ symbol and a } \frac{1}{2} \text{ coefficient.} \\ \rightsquigarrow^* \frac{1}{2} \left( \begin{array}{c} \text{Diagram of a circuit with two controls and one target, with a } \frac{1}{2} \text{ coefficient.} \\ + \quad \text{Diagram of a circuit with two controls and one target, with a } 0 \text{ coefficient.} \\ - \quad \text{Diagram of a circuit with two controls and one target, with a } 1 \text{ coefficient.} \end{array} \right) \end{array}$$

## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a phase shift of } \frac{\pi}{4} \text{ on the control wire.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Diagram of a 2-qubit gate with controls at the top and targets at the bottom. The first qubit has basis states } |1\rangle \text{ and } |0\rangle \text{ with arrows pointing down. The second qubit has basis states } |0\rangle \text{ and } |1\rangle \text{ with arrows pointing up. This is followed by a unitary operation indicated by a curly brace and a factor of } \frac{1}{2}. \end{array} \rightsquigarrow * \frac{1}{2} \begin{pmatrix} \text{Diagram of the same 2-qubit gate, but with the second qubit controls swapped (top) and targets swapped (bottom).} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a } \frac{1}{\sqrt{2}} \text{ coefficient.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Diagram of a multi-qubit gate with two controls and one target, with a } \frac{1}{2} \text{ coefficient.} \\ \rightsquigarrow * \frac{1}{2} \left( \begin{array}{c} \text{Diagram 1: Two controls (1 down, 0 down) and one target (0 down).} \\ \text{Diagram 2: Two controls (1 down, 0 down) and one target (1 down).} \end{array} - \begin{array}{c} \text{Diagram 3: Two controls (1 down, 0 down) and one target (0 down).} \\ \text{Diagram 4: Two controls (1 down, 0 down) and one target (1 down).} \end{array} \right) \end{array}$$

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Diagram of a CNOT gate with control on top and target on bottom, with a square symbol on the control wire.} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Diagram of a multi-controlled NOT gate with controls 1 and 0, and target on bottom, with square symbols on control wires.} \end{array} \rightsquigarrow^* \frac{1}{2\sqrt{2}} \left( \begin{array}{cccc} \text{Diagram 1: Control 1=1, Control 0=0, Target 0} & + & \text{Diagram 2: Control 1=1, Control 0=1, Target 1} & - \\ \text{Diagram 3: Control 1=0, Control 0=0, Target 1} & + & \text{Diagram 4: Control 1=0, Control 0=1, Target 1} & \end{array} \right)$$

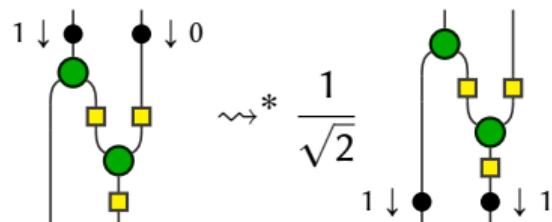
## Example

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{array}{c} \text{Initial State: } |1\rangle\langle 1| + |0\rangle\langle 0| \\ \text{Circuit: } \text{CNOT} \otimes \text{Identity} \\ \text{Result: } \frac{1}{2\sqrt{2}} \left( |11\rangle + |00\rangle - |10\rangle - |01\rangle \right) \end{array}$$

Diagram illustrating the decomposition of a CNOT gate into a superposition of four basis states. The initial state is  $|1\rangle\langle 1| + |0\rangle\langle 0|$ . The circuit consists of a CNOT gate followed by an identity gate. The result is a superposition of four states:  $\frac{1}{2\sqrt{2}} (|11\rangle + |00\rangle - |10\rangle - |01\rangle)$ . Red arrows point to the first two terms ( $|11\rangle + |00\rangle$ ), and blue arrows point to the last two terms ( $-|10\rangle - |01\rangle$ ).

$$\frac{1}{\sqrt{2}} \text{CNOT} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{Two qubits, CNOT gate} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



## Rewriting System

We define  $\rightsquigarrow$  as *exactly one diffusion rule followed by all possible collision rules until none applies.*

### Want to avoid:

- Having multiple tokens on the same edge that don't collide:



- Non-termination.

## Rewriting System

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### Two invariants:

- **Well-Formedness:** Avoid two tokens going in the same direction on a path.
- **Cycle-Balancedness:** Avoid tokens alone in cycles.

## Polarity in a Path

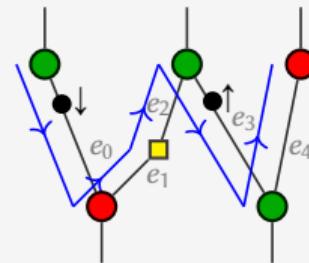
$p = (e_0, e_1, e_2, e_3, e_4)$  is an oriented path.

- If a token follows the path +1
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity

$$P(p, (e_0 \downarrow x)(e_3 \uparrow y)) = P(p, (e_0 \downarrow x)) + P(p, (e_3 \uparrow y)) = 0$$



## Well-Formed Token State

Given a ZX-Diagram and a Token State, it is **Well-Formed** if for every path  $p$  its Polarity  $\in \{-1, 0, 1\}$ .

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- Thm 5.3.12: Well-formed states cannot reach “bad configurations”.

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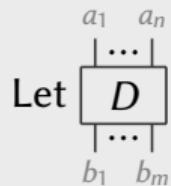
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Given a ZX-Diagram and a Token State, it is **Cycle-Balanced** if for every cycle  $c$  its Polarity = 0.

- Thm 5.3.16: Termination of well-formed, cycle-balanced token state.
- Prop 5.3.18: Local confluence of well-formed, cycle-balanced token state.

## Thm 5.3.25 (Simulation of Standard Interpretation)



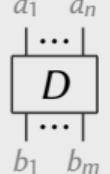
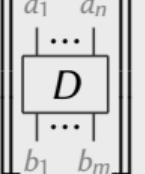
Let  $D$  a ZX-Diagram such that

$$\left[ \begin{array}{c} a_1 \quad a_n \\ \cdots \\ b_1 \quad b_m \end{array} \right] D \left[ \begin{array}{c} a_1 \quad a_n \\ \cdots \\ b_1 \quad b_m \end{array} \right] = \sum_{q=1}^{2^{m+n}} \lambda_q |y_{1,q} \dots y_{m,q}\rangle \langle x_{1,q} \dots x_{n,q}|$$

Let  $D =$

, consider  $t =$

## Thm 5.3.25 (Simulation of Standard Interpretation)

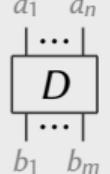
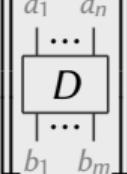
Let  a ZX-Diagram such that  =  $\sum_{q=1}^{2^{m+n}} \lambda_q |y_{1,q} \dots y_{m,q}\rangle \langle x_{1,q} \dots x_{n,q}|$

Let  $D = \begin{array}{c} \cdots \\ | \\ D' \text{ (CNOT-like gate)} \\ | \\ \cdots \end{array}$ , consider  $t = \begin{array}{c} \cdots \\ | \\ D' \text{ (CNOT-like gate with controls 0)} \\ | \\ \cdots \end{array} + \begin{array}{c} \cdots \\ | \\ D' \text{ (CNOT-like gate with controls 1)} \\ | \\ \cdots \end{array}$

Then

$$t \rightsquigarrow^* \sum_{q=1}^{2^{m+n}} \lambda_q \begin{array}{c} x_{1,q} \uparrow \bullet \cdots \bullet \uparrow x_{n,q} \\ | \\ D' \text{ (CNOT-like gate)} \\ | \\ y_{1,q} \downarrow \bullet \cdots \bullet \downarrow y_{m,q} \end{array}$$

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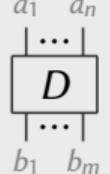
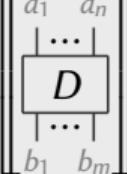
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# Towards a Curry-Howard Correspondence for Quantum Computation

$$[A] = \mu X. \mathbb{1} \oplus (A \otimes X)$$

l.f.p operator  
(Ch. 4, CSL'23)

Pairing  
(Ch. 5, MFCS'21)

Quantum Control  
(Ch. 6, Draft)

```
graph TD; A["l.f.p operator  
(Ch. 4, CSL'23)"] --> Eq; B["Pairing  
(Ch. 5, MFCS'21)"] --> Eq; C["Quantum Control  
(Ch. 6, Draft)"] --> Eq; Eq == "[A] = \mu X. \mathbb{1} \oplus (A \otimes X)"; style Eq fill:none,stroke:none; style A fill:none,stroke:none; style B fill:none,stroke:none; style C fill:none,stroke:none; class="highlight" style="color:blue; font-size:2em; font-weight:bold;">\otimes
```

# Towards a Curry-Howard Correspondence for Quantum Computation

$$[A] = \mu X. \mathbb{1} \oplus (A \otimes X)$$

l.f.p operator  
(Ch. 4, CSL'23)

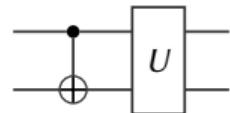
Pairing  
(Ch. 5, MFCS'21)

Quantum Control  
(Ch. 6, Draft)

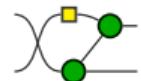
The diagram illustrates the Curry-Howard correspondence for quantum computation. It features a central equation  $[A] = \mu X. \mathbb{1} \oplus (A \otimes X)$ . Above the first term, 'l.f.p operator' and '(Ch. 4, CSL'23)' are shown. Above the second term, 'Pairing' and '(Ch. 5, MFCS'21)' are shown. Below the equation, 'Quantum Control' and '(Ch. 6, Draft)' are shown in red text. Curved arrows point from each of the top two sections down to the central equation, while a red curved arrow points from 'Quantum Control' up to the central equation.

# The Many-Worlds Calculus : When $\otimes$ meets $\oplus$

$\otimes$ -based languages

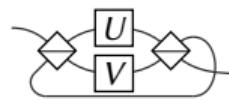


Quantum Circuits

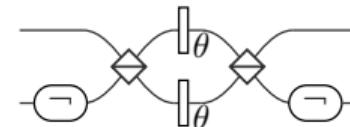


ZX-Calculus

$\oplus$ -based languages



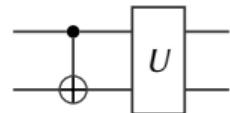
PBS-Calculus



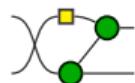
LOv-Calculus

# The Many-Worlds Calculus : When $\otimes$ meets $\oplus$

$\otimes$ -based languages

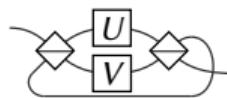


Quantum Circuits

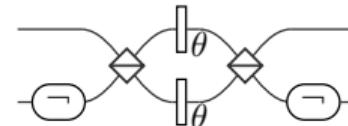


ZX-Calculus

$\oplus$ -based languages

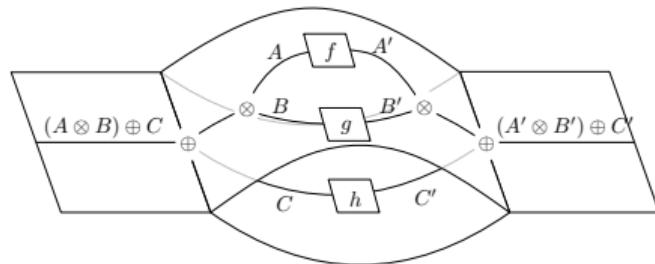


PBS-Calculus



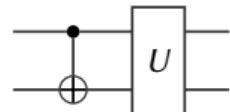
LOv-Calculus

The Many-Worlds Calculus

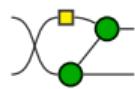


# The Many-Worlds Calculus : When $\otimes$ meets $\oplus$

## $\otimes$ -based languages

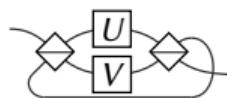


Quantum Circuits

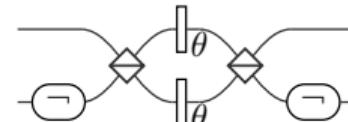


ZX-Calculus

## $\oplus$ -based languages

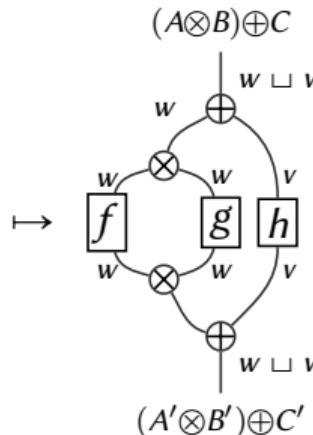
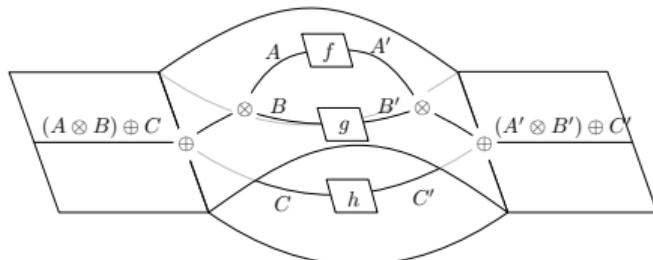


PBS-Calculus



LOv-Calculus

## The Many-Worlds Calculus

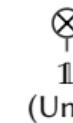
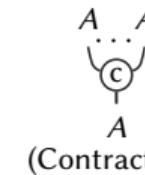
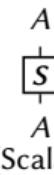


- Label **wires** with **worlds**.
- **Worlds** for naming **slices**.

---

## Generators

---

									
(Empty)	(Id)	(Swap)	(Cap)	(Cup)	$A \oplus B$ (Plus)	$A \otimes B$ (Tensor)	$\mathbb{1}$ (Unit)	$A \dots A$ A (Contraction)	$A$ $s$ A (Scalar)

---

## Generators

---

(Empty)	(Id)	(Swap)	(Cap)	(Cup)	$A \oplus B$ (Plus)	$A \otimes B$ (Tensor)	$\mathbb{1}$ (Unit)	$A \cdots A$ (Contraction)	$A$ (Scalar)

---

## Compositions

---

$$\begin{array}{ccc}
 \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \circ \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ \dots \\ | \\ D_2 \\ \dots \end{array} \\
 & & \\
 \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \parallel \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \quad \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array}
 \end{array}$$

---

## Generators

---

(Empty)	(Id)	(Swap)	(Cap)	(Cup)	$A \oplus B$ (Plus)	$A \otimes B$ (Tensor)	$\mathbb{1}$ (Unit)	$A \cdots A$ (Contraction)	$A$ (Scalar)

---

## Compositions

---

$$\begin{array}{ccc} \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \circ \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \\[10mm] \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \parallel \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \quad \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \end{array}$$

---

## Derived Constructors & Tokens

---

$$\begin{array}{c} A \oplus B \\ \oplus \\ A \quad B \end{array} := \begin{array}{c} A \quad B \\ \diagup \quad \diagdown \\ \oplus \end{array} \quad \begin{array}{c} A \oplus B \\ \oplus \\ A \quad B \end{array}$$

---

## Generators

---

(Empty)	(Id)	(Swap)	(Cap)	(Cup)	$A \oplus B$ (Plus)	$A \otimes B$ (Tensor)	$\mathbb{1}$ (Unit)	$A \cdots A$ (Contraction)	$A$ (Scalar)

---

## Compositions

---

$$\begin{array}{ccc} \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \circ \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ \dots \\ | \\ D_2 \\ \dots \end{array} \\[10mm] \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \parallel \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \quad \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \end{array}$$

---

## Derived Constructors & Tokens

---

$$\begin{array}{c} A \oplus B \\ \oplus \\ A \quad B \end{array} := \begin{array}{c} A \quad B \\ \diagup \quad \diagdown \\ \oplus \end{array}$$

$$\begin{array}{l} v ::= () \mid \langle s_1, s_2 \rangle \mid \text{inj}_\ell s \mid \text{inj}_r s \\ s ::= v \mid \text{skull} \end{array}$$

---

## Generators

---

(Empty)	(Id)	(Swap)	(Cap)	(Cup)	$A \oplus B$ (Plus)	$A \otimes B$ (Tensor)	$\mathbb{1}$ (Unit)	$A \cdots A$ (Contraction)	$A$ (Scalar)

---

## Compositions

---

$$\begin{array}{ccc} \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \circ \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ \dots \\ | \\ D_2 \\ \dots \end{array} \\[10mm] \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \parallel \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} & = & \begin{array}{c} \dots \\ | \\ D_1 \\ | \\ \dots \end{array} \quad \begin{array}{c} \dots \\ | \\ D_2 \\ | \\ \dots \end{array} \end{array}$$

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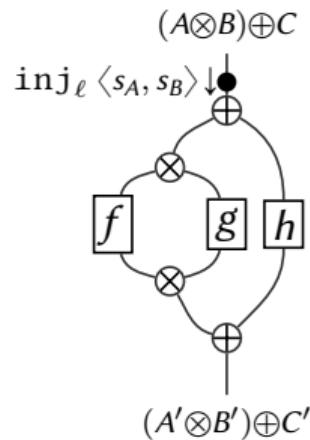
## Derived Constructors & Tokens

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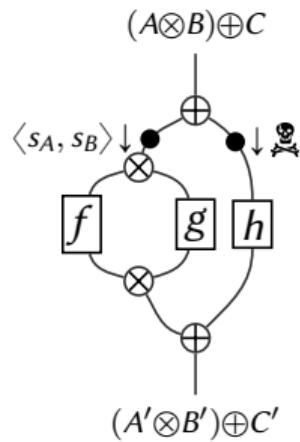
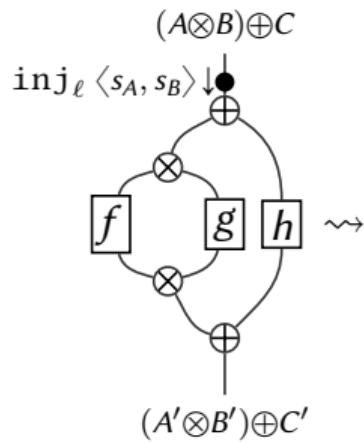
$$\begin{array}{c} A \oplus B \\ \oplus \\ A \quad B \end{array} := \begin{array}{c} \text{A} \quad \text{B} \\ \diagup \quad \diagdown \\ \oplus \end{array}$$

$$\begin{aligned} v &::= () \mid \langle s_1, s_2 \rangle \mid \mathbf{inj}_\ell s \mid \mathbf{inj}_r s \\ s &::= v \mid \text{skull} \\ \text{Token} &= (e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times s \end{aligned}$$

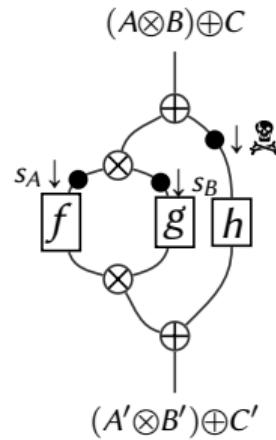
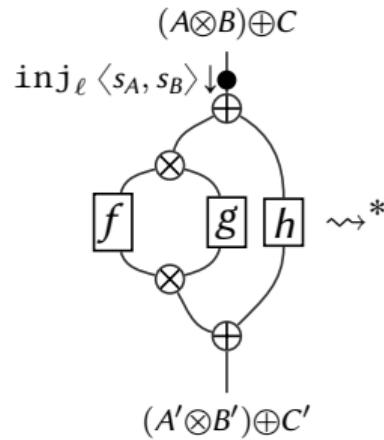
Assume  $s_A, s_B$  values of types  $A$  and  $B$ , then:



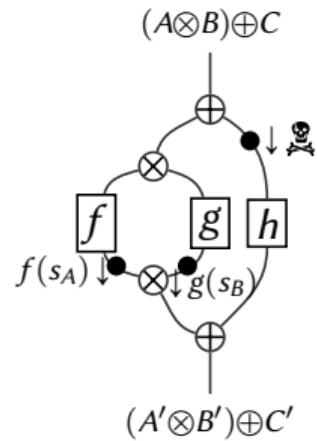
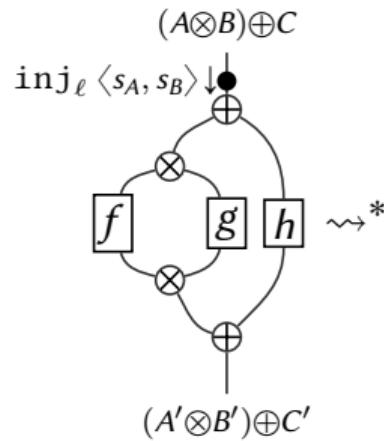
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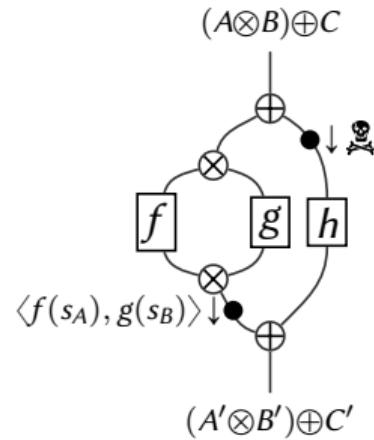
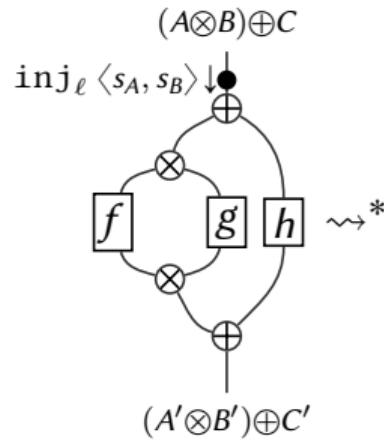
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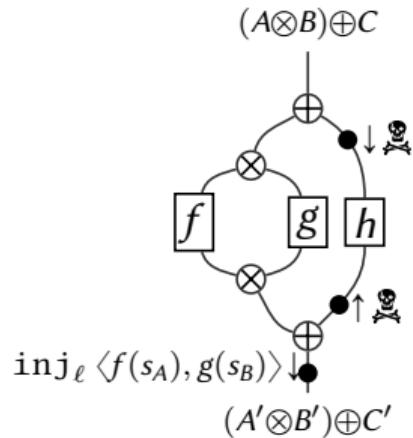
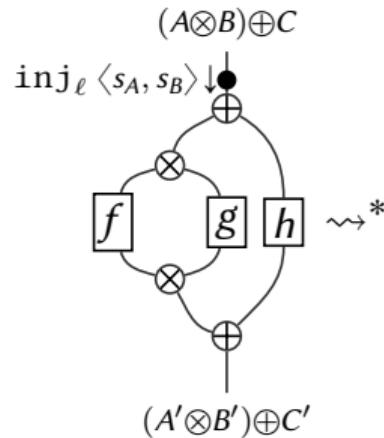
Assume  $s_A, s_B$  values of types  $A$  and  $B$ , then:



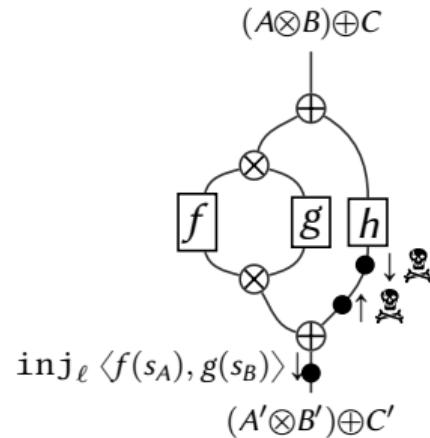
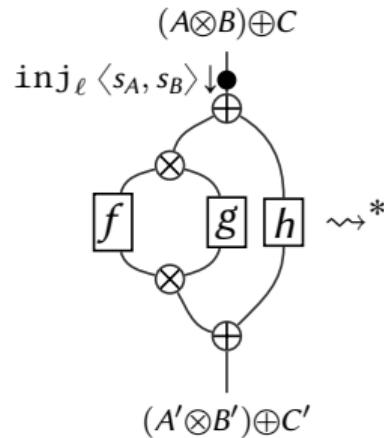
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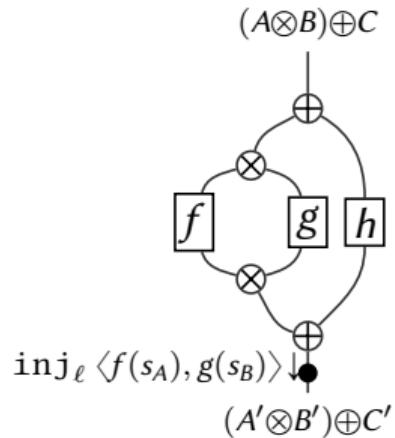
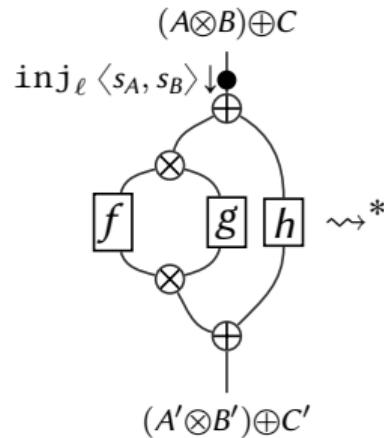
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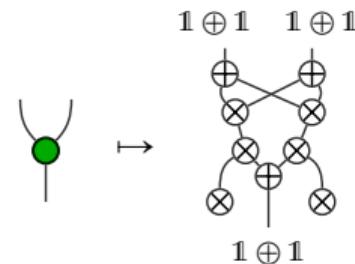


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## Results

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**Well-Formedness, Cycle-Balancedness** still holds.  
⇒ Confluence, Termination, Avoid bad configurations.

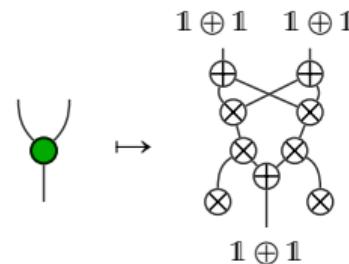


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## Results

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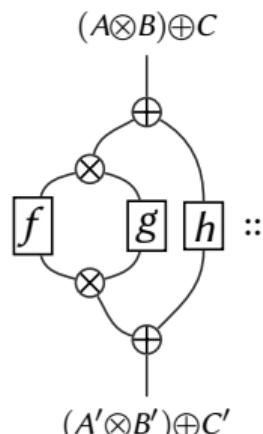
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## Quantum Control

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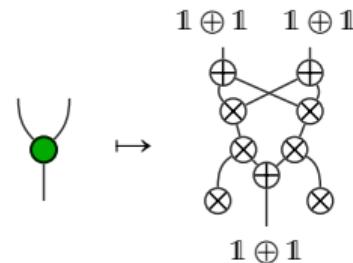


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## Results

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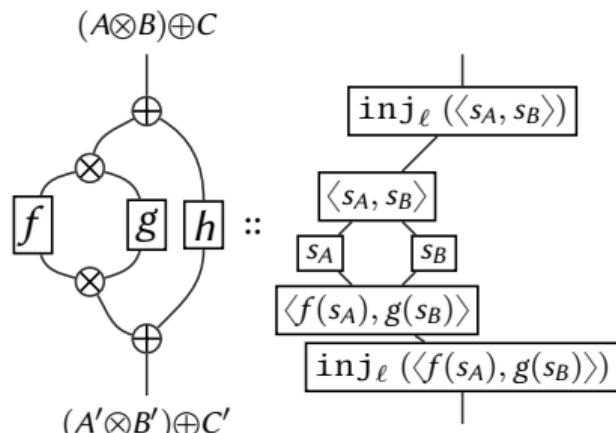
**Well-Formedness, Cycle-Balancedness** still holds.  
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## Quantum Control

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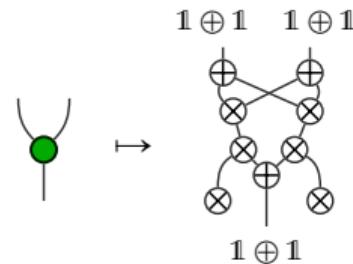


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## Results

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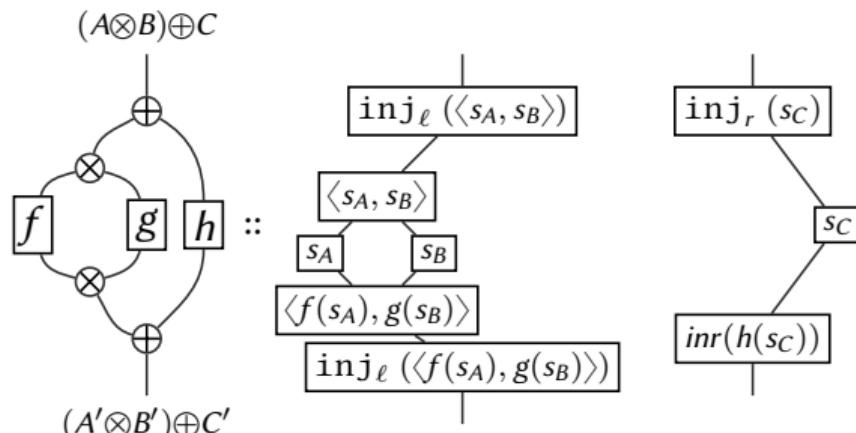
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## Quantum Control

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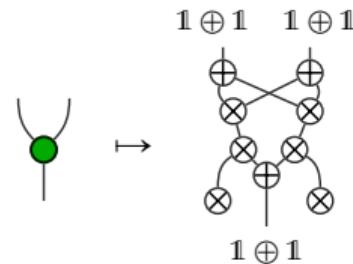


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## Results

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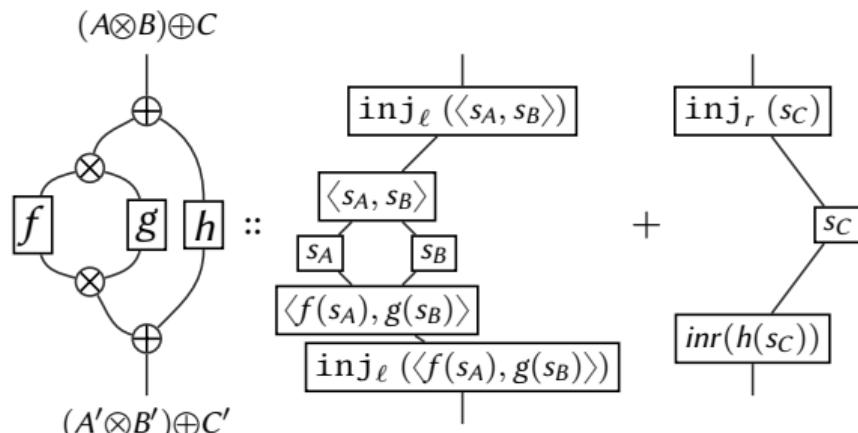
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## Quantum Control

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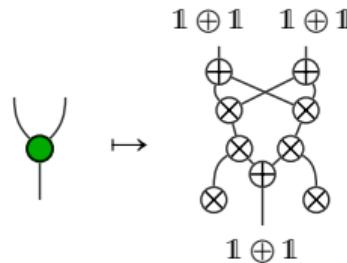


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## Results

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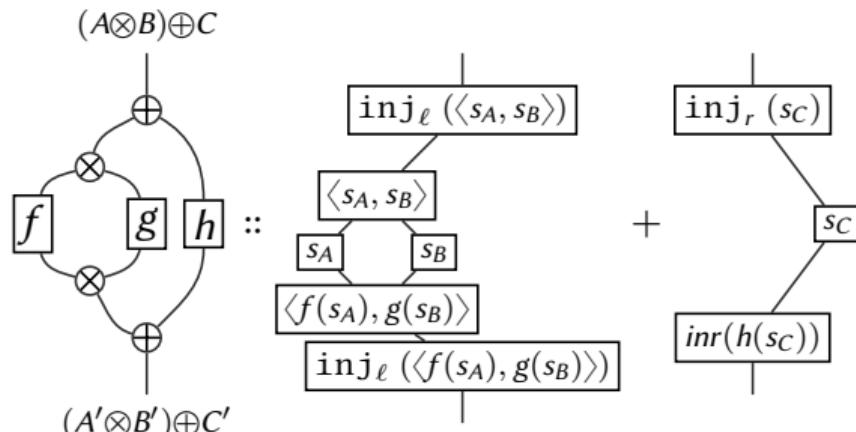
**Well-Formedness, Cycle-Balancedness** still holds.  
 ⇒ Confluence, Termination, Avoid bad configurations.




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## Quantum Control

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**Missing:** Inductive types.

# Towards a Curry-Howard Correspondence for Quantum Computation

$$[A] = \mu X. \mathbb{1} \oplus (A \otimes X)$$

l.f.p operator  
(Ch. 4, CSL'23)

Pairing  
(Ch. 5, MFCS'21)

Quantum Control  
(Ch. 6, Draft)

# Towards a Curry-Howard Correspondence for Quantum Computation

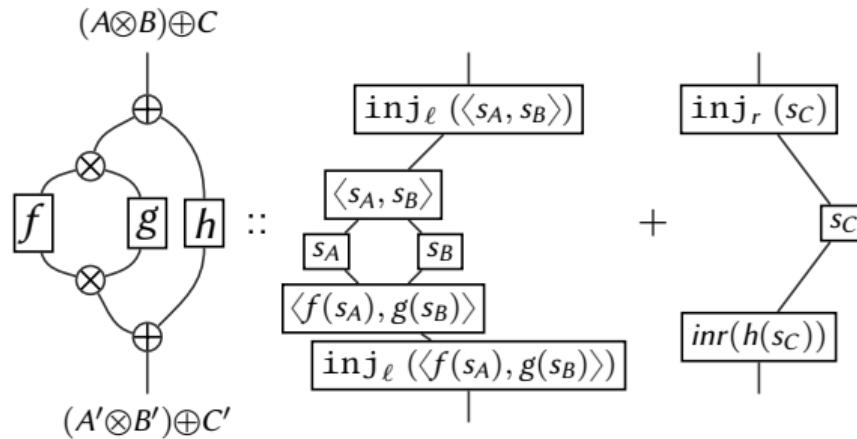
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Pairing  
(Ch. 5, MFCS'21)

Quantum Control  
(Ch. 6, Draft)

# A Syntax Term Language for the Many-Worlds Calculus



$$\left\{ \begin{array}{l} \inj_\ell (\langle x, y \rangle) \leftrightarrow \inj_\ell (\langle f x, g y \rangle) \\ \inj_r (z) \leftrightarrow \inj_r (h z) \end{array} \right\}$$

**Function from**  $(A \otimes B) \oplus C \leftrightarrow (A' \otimes B') \oplus C'$

(Base types)  $A, B ::= \mathbb{1} \mid A \oplus B \mid A \otimes B$

(Isos, first-order)  $T ::= A \leftrightarrow B$

(Values)  $v ::= x \mid () \mid \langle v_1, v_2 \rangle \mid \text{inj}_\ell v \mid \text{inj}_r v$

(Expressions)  $e ::= v \mid \text{let } x = \omega \ y \ \text{in } e$

(Isos)  $\omega ::= \{v_1 \leftrightarrow e_1 \mid \dots \mid v_n \leftrightarrow e_n\}$

(Base types)  $A, B ::= \mathbb{1} \mid A \oplus B \mid A \otimes B \mid \mu X.A \mid X$

(Isos, first-order)  $T ::= A \leftrightarrow B$

(Values)  $v ::= x \mid () \mid \langle v_1, v_2 \rangle \mid \text{inj}_\ell v \mid \text{inj}_r v \mid \text{fold } e$

(Expressions)  $e ::= v \mid \text{let } x = \omega \ y \ \text{in } e$

(Isos)  $\omega ::= \{v_1 \leftrightarrow e_1 \mid \dots \mid v_n \leftrightarrow e_n\} \mid \text{fix } f.\omega \mid f$

$$\text{map}(\omega) = \text{fix } f. \left\{ \begin{array}{l} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega h) :: (f t) \end{array} \right\} : [A] \leftrightarrow [B]$$

$$[ ] = \text{fold} (\text{inj}_\ell ()) \quad h :: t = \text{fold} (\text{inj}_r (\langle h, t \rangle))$$

$\mu\text{MALL}^\infty = \text{MALL} + \mu.$

$$\frac{\Delta, A[X \leftarrow \mu X.A] \vdash B}{\Delta, \mu X.A \vdash B} \mu_L \quad \frac{\Delta \vdash A[X \leftarrow \mu X.A]}{\Delta \vdash \mu X.A} \mu_R$$

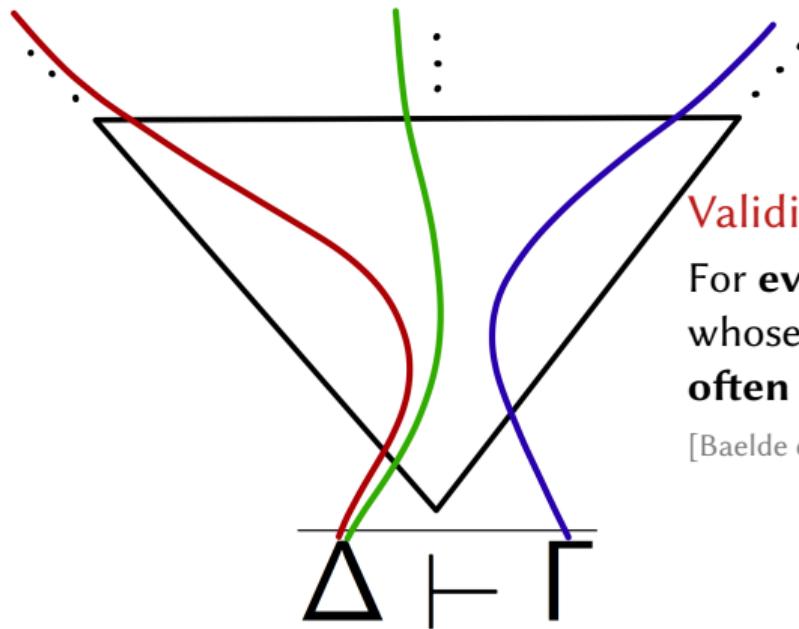
$$\mu\text{MALL}^\infty = \text{MALL} + \mu.$$

$$\frac{\Delta, A[X \leftarrow \mu X.A] \vdash B}{\Delta, \mu X.A \vdash B} \mu_L \quad \frac{\Delta \vdash A[X \leftarrow \mu X.A]}{\Delta \vdash \mu X.A} \mu_R$$

**Non-wellfounded proofs:**

$$\frac{\vdots \quad \vdots}{\frac{\mu X.X \vdash F \quad \vdash \mu X.X}{\vdash F}} \mu \quad \frac{\mu X.X \vdash F \quad \vdash \mu X.X}{\vdash F} \text{cut}$$

There is a need for a **validity criterion** on derivations.



## Validity Condition

For **every infinite branch**, there is a **thread** whose **smallest formula** that occurs **infinitely often** is a  $\mu$  formula on the **left**.

[Baelde et al., 2016, 2022]

Let us take the  $\text{map}(\omega)$  functions on lists.

$$\mathbf{fix} \ f. \ \left\{ \begin{array}{l} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega \ h) :: (f \ t) \end{array} \right\} : [A] \leftrightarrow [B]$$

Send it to a derivation  $\text{proof}(\text{map}(\omega)) : [A] \vdash [B]$ .

# From Derivations To Proofs - Example

$$\frac{\frac{\vdash [B]}{1 \vdash [B]} \mathbb{1}_L \quad \frac{A, [A] \vdash [B]}{A \otimes [A] \vdash [B]} \otimes_L}{\frac{\mathbb{1} \oplus (A \otimes [A]) \vdash [B]}{[A] \vdash [B]}} \oplus_L \mu_L$$

**fix**  $f$ .  $\left\{ \begin{array}{l} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega h) :: (f t) \end{array} \right\}$

# From Derivations To Proofs - Example

$$\frac{\frac{\frac{\frac{\vdash \mathbb{1} \quad \mathbb{1}_R}{\vdash \mathbb{1} \oplus (B \otimes [A]) \quad \oplus_R^1}}{\vdash \mathbb{1} \oplus (B \otimes [A]) \quad \mu_R}}{\vdash [B] \quad \mathbb{1}_L} \quad \frac{A, [A] \vdash [B]}{A \otimes [A] \vdash [B]} \quad \frac{\otimes_L}{\oplus_L}}$$
$$\frac{\mathbb{1} \oplus (A \otimes [A]) \vdash [B] \quad \mu_L}{[A] \vdash [B]}$$

**fix**  $f$ .  $\left\{ \begin{array}{c} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega h) :: (f t) \end{array} \right\}$

# From Derivations To Proofs - Example

$$\begin{array}{c}
 \frac{\vdash \mathbf{1} \quad \mathbf{1}_R}{\vdash \mathbf{1} \oplus (B \otimes [A])} \oplus_R^1 \quad \frac{A, [A] \vdash B \otimes [B]}{A, [A] \vdash \mathbf{1} \oplus (B \otimes [B])} \oplus_R^2 \\
 \frac{}{\vdash [B] \quad \mathbf{1}_L} \mu_R \quad \frac{A, [A] \vdash [B]}{A \otimes [A] \vdash [B]} \otimes_L \\
 \hline
 \frac{\vdash \mathbf{1} \oplus (A \otimes [A]) \vdash [B] \quad \mathbf{1}_L}{[A] \vdash [B]} \mu_L
 \end{array}$$

**fix**  $f$ .  $\left\{ \begin{array}{l} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega h) :: (f t) \end{array} \right\}$

# From Derivations To Proofs - Example

$$\begin{array}{c}
 \frac{\omega}{A \vdash B} \\
 \frac{}{\vdash \mathbb{1} \quad \mathbb{1}_R} \quad \frac{}{\vdash \mathbb{1} \oplus (B \otimes [A]) \quad \mu_R} \quad \frac{\omega}{A \vdash B} \quad \frac{}{A, [A] \vdash B \otimes [B]} \otimes_R \\
 \frac{}{\vdash \mathbb{1} \oplus (B \otimes [A]) \quad \mu_R} \quad \frac{}{A, [A] \vdash \mathbb{1} \oplus (B \otimes [B]) \quad \mu_R} \quad \frac{}{A, [A] \vdash [B]} \quad \frac{}{A \otimes [A] \vdash [B]} \otimes_L \\
 \frac{\vdash [B] \quad \mathbb{1}_L}{\mathbb{1} \vdash [B]} \quad \frac{}{A, [A] \vdash [B]} \quad \frac{}{A \otimes [A] \vdash [B]} \oplus_L \\
 \hline
 \frac{\vdash [B] \quad \mathbb{1}_L \quad A, [A] \vdash [B] \quad A \otimes [A] \vdash [B]}{\vdash \mathbb{1} \oplus (A \otimes [A]) \vdash [B]} \quad \mu_L \\
 \frac{}{[A] \vdash [B]}
 \end{array}$$

**fix**  $f$ .  $\left\{ \begin{array}{l} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega \ h) :: (f \ t) \end{array} \right\}$

# From Derivations To Proofs - Example

$$\frac{\frac{\frac{\frac{\frac{\vdash \mathbb{1}}{\vdash \mathbb{1} \ 1_R}{\oplus_R^1}}{\vdash \mathbb{1} \oplus (B \otimes [A])}{\mu_R}}{\vdash [B]} \ 1_L}{\vdash \mathbb{1} \oplus (A \otimes [A]) \vdash [B]}{\mu_L}}{\frac{\frac{\frac{\omega}{A \vdash B}{\vdash [B]}{\vdots}}{A, [A] \vdash B \otimes [B]}{\otimes_R}}{\frac{\frac{A, [A] \vdash B \otimes [B]}{A, [A] \vdash \mathbb{1} \oplus (B \otimes [B])}{\oplus_R^2}}{\frac{\frac{A, [A] \vdash [B]}{A \otimes [A] \vdash [B]}{\otimes_L}}{\frac{\frac{A \otimes [A] \vdash [B]}{\mathbb{1} \oplus (A \otimes [A]) \vdash [B]}{\oplus_L}}{\vdash [A] \vdash [B]}}{\mu_L}}{\mu_R}}{\mu_R}}$$

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# From Derivations To Proofs - Example

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$\omega$        $\vdots$   
 $\dfrac{}{A \vdash B}$      $\dfrac{}{[A] \vdash [B]}$   
 $\dfrac{}{A, [A] \vdash B \otimes [B]}$   $\otimes_R$   
 $\dfrac{}{A, [A] \vdash \mathbb{1} \oplus (B \otimes [B])}$   $\oplus_R^2$   
 $\dfrac{}{A, [A] \vdash [B]} \otimes_L$   
 $\dfrac{}{A \otimes [A] \vdash [B]} \oplus_L$

**fix**  $f$ .  $\left\{ \begin{array}{c} [ ] \leftrightarrow [ ] \\ h :: t \leftrightarrow (\omega \ h) :: (f \ t) \end{array} \right\}$

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## Language

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Iso have an well-typed inverse. Lemma 4.2.11:  $\omega : A \leftrightarrow B$  then  $\omega^{-1} : B \leftrightarrow A$ .

Iso are isomorphisms. Thm 4.2.13:  $\omega \circ \omega^{-1} = \text{Id}$ .

Subject Reduction & Progress. Lemma 4.2.18 & 4.2.19.

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## Curry-Howard

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Proof Validity. Thm 4.4.20: If  $\omega : A \leftrightarrow B$  then  $\text{proof}(\omega) : A \vdash B$  is a proof.

Cut-Elimination Simulation. Thm 4.4.29: If  $t \rightarrow t'$  then  $\text{proof}(t) \rightarrow \text{proof}(t')$ .

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## ZX Token Machine: MFCS'21

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- **With:** Benoît Valiron, Renaud Vilmart.
- **What:** Study of Token-Based semantics for the ZX-Calculus.
- **Why:** First approach for control quantum, led to the Many-Worlds Calculus.

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## Isomorphisms & $\mu$ MALL: CSL'23

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- **With:** Alexis Saurin, Benoît Valiron.
- **What:** Linear, reversible programming language with inductive types.
- **Why:** First step towards quantum inductive types.

## Future Work:

- Quantum Programming Language with Inductive Types.
- Many-Worlds with Inductive Types.
- Relation between the two ?

## Future Work:

- Quantum Programming Language with Inductive Types.
- Many-Worlds with Inductive Types.
- Relation between the two ?

## First step:

- Remove inductive types.
- Add superposition of expressions.
- Translate iso as Many-Worlds diagrams.
- Thm 6.9.14 : Soudness.